SELF-SIMILARITY OF A CONVECTIVE HEAT-TRANSFER PROBLEM

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We study the system of differential equations for the fluid velocity and fluid temperature in a two-dimensional channel and also in a circular tube in a region of stabilized heat transfer. On the tube walls we specify boundary conditions of the second kind; we assume that the viscosity depends exponentially on the temperature. We consider the conditions under which one-dimensional nonisothermal flows arise.

1. The majority of problems in hydrodynamics and heat transfer concerned with fluid flow in tubes is complicated by the fact that a transverse velocity component appears due to the change in viscosity along the tube length, (see [1, 2]). It is of interest to find a class of nonisothermal flows which remain one-dimensional in spite of the longitudinal viscosity gradient. The problem concerning heat transfer and hydraulic resistance considered in [3] is an example of such a class; our aim in this paper is to make explicit some common regularities inherent in such flows.

We assume that well beyond the entrance to a cylindrical tube, with a sufficiently smooth contour for its cross section, a flow regime is established with the following properties: 1) the velocity field in the direction of the x axis is one-dimensional, i.e., $v_x = v(y, z)$; 2) the temperature profiles at each section x = const are similar to one another, i.e., $t(x, y, z) = Ax + t_1(y, z)$ $(A \neq 0)$.

We assume, further, that the viscosity of the flow may be approximated by an exponential function of the temperature:

$$\mu / \mu_0 = e^{-\beta(t-t_0)} \qquad (\mu_0,\ \beta - \text{const}) . \tag{1.1}$$

When dissipation is not taken into account the system of equations of motion and heat transfer has the form

$$\begin{gathered}
\frac{\partial P}{\partial X} = \frac{\partial}{\partial Y}\left[e^{-\alpha(X+\theta)}\frac{\partial U}{\partial Y}\right] + \frac{\partial}{\partial Z}\left[e^{-\alpha(X+\theta)}\frac{\partial U}{\partial Z}\right] \\
\frac{\partial P}{\partial Y} = -\alpha e^{-\alpha(X+\theta)}\frac{\partial U}{\partial Y}, \qquad \frac{\partial P}{\partial Z} = -\alpha e^{-\alpha(X+\theta)}\frac{\partial U}{\partial Z} \\
\Delta\theta = \mathrm{Pe}\, U \qquad (\Delta = \partial^2/\partial Y^2 + \partial^2/\partial Z^2) \\
X = x/l, \qquad Y = y/l, \qquad Z = z/l, \qquad U = v/U_0 \\
\theta = t_1/Al, \qquad P = pl/\mu_0 U_0, \qquad \mathrm{Pe} = U_0 l/a, \qquad \alpha = \beta A l .
\end{gathered} \tag{1.2}$$

Here p is the fluid pressure; a is the coefficient of thermal diffusivity; U_0 is the mean outflow rate of the fluid; and l is the characteristic dimension of the cross section.

We can eliminate the velocity from the first three equations of the system (1.2) and so obtain an equation for the pressure:

$$\Delta P = -\alpha \partial P / \partial X . \tag{1.3}$$

Separating out the exponential factor, $P(X, Y, Z) = F(Y, Z)\exp(-\alpha X)$, we have the following equation for the function $F(Y, Z)$:

$$\Delta F = \alpha^2 F . \tag{1.4}$$

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From the second and third equations of the system we obtain

$$\frac{D(F, U)}{D(Y, Z)} = \frac{D(\theta, U)}{D(Y, Z)} \,. \tag{1.5}$$

Since the fluid speed along the boundary of the flow region vanishes, it follows from Eq. (1.5) that the function F(Y, Z) is constant along the perimeter of the contour and on the isotherms. In other words, for the existence of a nonisothermal flow with the properties 1) and 2) it is necessary that the pressure and speed of the fluid be constant on the isotherms of the flow.

Transforming the system (1.2) with the aid of Eqs. (1.3) and (1.4), we reduce it to the form

$$\frac{dU}{dF} = -\frac{1}{\alpha} e^{\alpha\theta}, \quad \frac{d^2\theta}{dF^2} |\operatorname{grad} F|^2 - \alpha^2 F \frac{d\theta}{dF} = \operatorname{Pe} U \,. \tag{1.6}$$

In order that the second equation of the system (1.6) be satisfied, it is necessary to assume that

$$|\operatorname{grad} F|^2 = \xi(F) \tag{1.7}$$

since the rest of the terms depend only on F.

Thus, for the determination of F(Y, Z) we have Eq. (1.4), with the condition that F(Y, Z) be constant on the contour of the tube cross section, and also the additional condition (1.7). The solution of this problem is well known [4, 5]: the boundary contour and the flow isotherms must be curves of constant curvature.

On the basis of these results we can assert that a one-dimensional nonisothermal fluid flow with an exponential dependence of the viscosity on the temperature, with heat transfer by convection taken into account, does, in fact, exist in a two-dimensional channel, in a circular tube, and between coaxial circular tubes.

2. For flow of a fluid in a two-dimensional channel and in a circular tube Eq. (1.4) has an exact solution relative to which we easily obtain the following systems of equations:

For the two-dimensional channel

$$\begin{gathered} \frac{dU}{dY} = -(B \operatorname{ch} \alpha Y + C \operatorname{sh} \alpha Y) e^{\alpha\theta}, \quad \frac{d^2\theta}{dY^2} = \operatorname{Pe} U \\ P(X, Y) = (C \operatorname{ch} \alpha Y + B \operatorname{sh} \alpha Y) e^{-\alpha X} \,. \end{gathered} \tag{2.1}$$

For the circular tube

$$\begin{gathered} \frac{dU}{dR} = (BI_1(\alpha R) + CK_1(\alpha R)) e^{\alpha\theta} \\ \frac{d^2\theta}{dR^2} + \frac{1}{R}\frac{d\theta}{dR} = \operatorname{Pe} U \qquad (R = \sqrt{X^2 + Y^2}) \\ P(X, R) = (BI_0(\alpha R) + CK_0(\alpha R)) e^{-\alpha X} \,. \end{gathered} \tag{2.2}$$

Here $I_0(\alpha R)$, $I_1(\alpha R)$, $K_0(\alpha R)$, $K_1(\alpha R)$ are Bessel functions of an imaginary argument. We note several consequences of the systems of equations (2.1) and (2.2).

For the frictional flow stress in the two-dimensional channel and in the circular tube we have, respectively,

$$G / G_w = \operatorname{sh} \alpha Y / \operatorname{sh} \alpha, \quad G / G_w = I_1(\alpha R) / I_1(\alpha) \tag{2.3}$$

where G_w is the frictional stress at the wall.

It is interesting to note that the relations (2.3) do not contain the Peclet parameter Pe and that they are completely determined by specifying the parameter α, which is proportional to the thermal loading on the wall. When $\alpha \to 0$ the relations (2.3) reduce to well-known linear relationships.

When α is small, the assumption made in [2] concerning replacing the actual tangential stress by a linear stress is well justified if the thermal wall loading is not very large.

For the system of equations (2.1) we can display a particular solution having a physical meaning.* When B=0, |C| =−C, it is easy to verify directly that the functions

$$\theta = \frac{1}{\alpha} \ln \frac{6\alpha^2}{\mathrm{Pe}\,|C|\,\mathrm{ch}^3 \alpha Y}, \quad U = -\frac{3\alpha}{\mathrm{Pe}\,\mathrm{ch}^2 \alpha Y}$$

satisfy the equations (2.1).

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